

# Fine Structure of Avalanches in the Abelian Sandpile Model

Amir Abdolvand and Afshin Montakhab\*

*Physics Department, College of Science, Shiraz University, Shiraz 71454, Iran.*

We study the two-dimensional Abelian Sandpile Model on a square lattice of linear size  $L$ . We introduce the notion of avalanche's fine structure and compare the behavior of avalanches and waves of toppling. We show that according to the degree of complexity in the fine structure of avalanches, which is a direct consequence of the intricate superposition of the boundaries of successive waves, avalanches fall into two different categories. We propose scaling ansatz for these avalanche types and verify them numerically. We find that while the first type of avalanches has a simple scaling behavior, the second (complex) type is characterized by an avalanche-size dependent scaling exponent. This provides a framework within which one can understand the failure of a consistent scaling behavior in this model.

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## I. INTRODUCTION

Bak, Tang, and Wiesenfeld (BTW) introduced the notion of Self-Organized Criticality (SOC) [1,2] as a possible mechanism for the generic emergence of spatial and temporal power law correlations. To elucidate the concept of SOC, they introduced a cellular automaton known as sandpile model which is an example of slowly driven, spatially extended, dissipative dynamical system [3]. The generality inherent in the basic notions of SOC has led to its successful application in various problems in physics as well as biology [4,5,6,7]. The common characteristics of all these systems is that at the self-organized critical state the microscopic details of the system are shadowed by the collective behavior of the individual constituents of the system. Due to the simplicity of local dynamical rules, and the ease with which they are implemented on a computer, many different models exhibiting SOC have been introduced and studied by various authors [8,9]. However, the prototypical sandpile model of SOC and a variant of it known as the Abelian Sandpile Model (ASM) [10], has resisted many clever efforts in fully understanding its dynamical behavior [11,12,13,14,15]. This is despite the fact that the analytical tractability of this model, which enables one to evaluate exactly many of its static properties [10,16,17,18], had provided hope for a better understanding of its dynamical properties. Currently, a complete description of the dynamical properties of the ASM is still missing [19].

To highlight this important point, we concentrate on the event-size (avalanche) distribution function. In order to check the assumption that the characteristic properties of avalanches in the critical state are described by scale free distribution functions with cutoffs limited only by the finite size effects, BTW proposed a simple picture of

finite-size scaling (FSS) in analogy with more traditional critical phenomena [3,20,21]. Although, it is now generally accepted that simple FSS picture fails in describing the scaling behavior of avalanches in the BTW model, the reasons suggested for this inconsistency seem to be very different [22,23,24,25,26]. For example, in a large-scale simulation, Drossel [26] explained the deviations from pure power law behavior by dividing avalanches into dissipative and non-dissipative avalanches. However, such attempts which aim at relating the deviations from pure power law behavior to the finite-size (or boundary) effects seem to be problematic. In fact, it is shown by Ktitarov et. al [27], that by reducing such effects one still observes the aforementioned deviations from the simple power law behavior.

Due to the complex spatiotemporal behavior of avalanches, it is reasonable to decompose them into more elementary objects. As is shown in Ref. [28], due to the Abelian property of the model which admits an interchangeable order in the relaxation of local instabilities, one can consider an avalanche as a composition of a series of (global) instabilities which are referred to as waves of topplings. The inconsistencies in our understanding of the dynamical behavior of avalanches reveal themselves more clearly when we consider that an ensemble of waves behaves simply and obeys FSS ansatz [27]. This is in contrast with the complex behavior of avalanches. That is, avalanches which are *presumably* simple composition of waves do not obey FSS. In view of the aforementioned points, the following main questions arise: While the time evolution of an avalanche differs only in the order of the relaxation of local instabilities with that of a wave, what makes the two events have such different scaling behaviors? Moreover, beside peculiarities imposed by boundaries of the system and finite size effects, what possible mechanism, probably inherent in the dynamical behavior of an avalanche itself, might be responsible for the observed complexity in the scaling behavior of avalanches?

In general, one expects to answer these questions by considering the effect of "nontrivial composition" of cor-

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\*Corresponding author. Tel.: +98-711-2284609; Fax: +98-711-2284594  
E-mail address: montakhab@shirazu.ac.ir

related waves [29,30]. However, to obtain a clear quantitative picture one needs to clarify beforehand what exactly “nontrivial composition” means, how this nontriviality is related to the complexity in avalanche dynamics, and last but not least, how this is related to the failure of simple scaling picture [31]. In the present work, we analyze the spatiotemporal structure of avalanches in order to investigate the above issues. The bulk of an avalanche consists of sites which have toppled as well as their nearest neighbors. This might lead one to naively believe that the bulk of an avalanche consists of sites which are unchanged, i.e. recurrent, where only sites on the boundary of an avalanche change their states (dynamical variable). However, avalanches can have complex internal structures. As waves of topplings occur during an avalanche, the boundaries of these waves could interact with each other leading to a complex internal (bulk) structure consisting of both recurrent and non-recurrent states, see for example Fig. 1(b). Using these facts, we classify avalanches into two classes, simple and complex, and investigate their scaling behavior. We show that different classes of avalanches have distinctly different scaling behavior. In particular, while the scaling behavior of the first type is observed to be independent of avalanche-size, in the second type an avalanche-size dependent scaling exponent is found. We therefore argue this to be the main cause of inconsistent scaling behavior in avalanche statistics of the ASM.

The present article is organized as follows: In Section II, we give a brief review of the basic concepts and definitions of the ASM. In Section III, we analyze the spatial structures of avalanches and introduce the idea of the avalanche’s fine structure. In Section IV, we compare the behavior of avalanches and waves of topplings and argue that according to the degree of complexity in their fine structures, avalanches fall into two different categories, type  $\alpha$  and type  $\beta$ . In Section V, we propose scaling ansatz for the two types of avalanches and verify them numerically. Finally, Section VI is devoted to a short summary and outlook.

## II. TWO-DIMENSIONAL ASM AND THE WAVE PICTURE OF EVOLUTION

The two-dimensional ASM is a cellular automaton defined on a square lattice of linear size  $L$ . To every point of the lattice there corresponds an integer dynamical variable  $h_i$ , which in the language of sandpiles represents the height of the column of sand at the  $i^{th}$  site. To simulate external drive, the system is perturbed by increasing the dynamical variable of a randomly chosen site by one,

$$h_i \rightarrow h_i + 1. \quad (1)$$

This can be interpreted as an increase in the local value of height, energy, pressure, etc. A site is considered unstable if its dynamical variable exceeds a predefined threshold value ( $h_i > h_c$ ). An unstable site then topples, upon

which its dynamical variable is decreased by 4, whilst each of its four nearest neighbors ( $nn$ ) receive one unit of energy:

$$h_i \rightarrow h_i - 4, \quad (2)$$

$$h_{nn} \rightarrow h_{nn} + 1. \quad (3)$$

In turn, through the relaxation processes, see Eqs.(3), the neighboring sites may become unstable themselves, leading to a series of instabilities the sum of which is referred to as an avalanche. Since the local dynamical rules are conservative, the dissipation can take place only at the boundary of the system. Here we will use open boundary conditions where, if an unstable site is on the boundary, one or more grains of sand will leave the system.

The method generally used during the relaxation of an avalanche is the parallel updating method, where all unstable sites are relaxed simultaneously during an instant in the relaxation process. Due to the Abelian nature of the model, the order of the toppling during an avalanche does not affect the final state [10]. Therefore, beside the parallel method of updating, it is possible to perform the relaxation process by a succession of waves. There is a simple dynamical procedure leading to such a decomposition [28]. In this method, we relax the seeding site, say  $i$ , after its first instability. This may cause further instabilities in the neighboring sites. We then relax all other unstable sites, except the seeding site  $i$ . The set of all toppled sites during this process forms the first wave of toppling. If, after the termination of the first wave, the seeding site is still unstable, we repeat the above procedure obtaining the second, third, etc. waves of topplings. This procedure continues until all sites are stable again. Therefore, one can consider the relaxation process of an avalanche as a sequence of waves of topplings, all of which originate from the seeding site. While in the former method waves overlap in time, by decomposing an avalanche to a sequence of waves, only one wave propagates at a time. This method of updating enables us to view the time evolution of the model in an ensemble of waves.

It can be shown both analytically and numerically [27] that the scaling property of waves is simple and obeys FSS. However, avalanches, which might naively be considered as a simple sum of waves, do not have simple scaling behavior and in fact do not obey FSS [22,23,27]. We believe that the reason for this discrepancy is found in the fine structure of avalanches which is due to wave boundary interactions.

In the present work we show that the boundaries of waves making up a given avalanche could interact with each other. This interaction of boundaries may lead to a complex spatial structure within an avalanche bulk structure which substantially changes the dynamical properties of the avalanche, including its scaling behavior. To show this, we divide avalanches into two distinct classes based on the complexity in their (internal) fine structure: simple (type  $\alpha$ ) avalanches that behave much like waves,

and complex (type  $\beta$ ) avalanches which have distinct different, size-dependent scaling behavior.

### III. FINE STRUCTURE OF AN AVALANCHI

The area  $a$  of a relaxation process  $R$  (avalanche wave) is defined as the number of distinct sites toppled during that process. In general, this area can be divided into two different structures which make up the fine structure of an avalanche. The first structure consists of those sites for which the corresponding dynamical variables have remained unchanged before and after the relaxation event. In fact, these sites may have toppled once or more during the relaxation event. But due to the balance between the outflow (Eq. (2)), and inflow (Eq. (3)) of particles through the relaxation of their near neighbors, their states remain unchanged after the termination of the relaxation process. The second structure consists of sites whose dynamical variables have changed as a result of a relaxation event.

Let us denote by  $|h_i, t\rangle$ ,  $i = 1, \dots, N$  the single site microstates of the system at time  $t$ , where  $t$  denotes the macroscopic time scale of the system. We denote by  $|h_i, t\rangle_I$  and  $|h_i, t\rangle_F$ , the initial and final microstate of the  $i^{th}$  site before and after the occurrence of the  $t^{th}$  relaxation process. A Recurrent Macrostate ( $\mathcal{RM}$ ) of the system consists of those sites for which the corresponding single site microstates satisfy the following relation,

$$i \in (\mathcal{RM}) \Leftrightarrow |h_i, t\rangle_F = |h_i, t\rangle_I \wedge i \in R \quad (4)$$

We say that the site  $i$  belongs to the relaxation process  $R$ ,  $i \in R$ , if and only if  $i$  has toppled during that process.  $R$  is the region affected by the process.

In a similar way, one can define the Non-recurrent Macrostate ( $\mathcal{NM}$ ) of a relaxation process as a collection of sites for which the corresponding single site microstates satisfy the relation,

$$i \in (\mathcal{NM}) \Leftrightarrow |h_i, t\rangle_F \neq |h_i, t\rangle_I \wedge i \in R \quad (5)$$

According to this definition those sites that form the exterior boundary of a relaxation process, whose states change without toppling, are excluded from the corresponding  $\mathcal{NM}$  structure.

Let  $n_{\mathcal{RM}(\mathcal{NM})}$  be the size of a recurrent (non-recurrent) macrostate, i.e. the total number of single site recurrent (non-recurrent) microstates. Then for the area of the avalanche we have,

$$a = n_{\mathcal{RM}} + n_{\mathcal{NM}}. \quad (6)$$

When the internal fine structure of an avalanche is made up of multiple, spatially contained waves, the  $\mathcal{NM}$  constitutes a thin layer on the boundary of each wave, while the bulk of the relaxation process is mainly made up of a  $\mathcal{RM}$  structure. On the other hand, when an avalanche is composed of multiple, interpenetrating waves, this simple

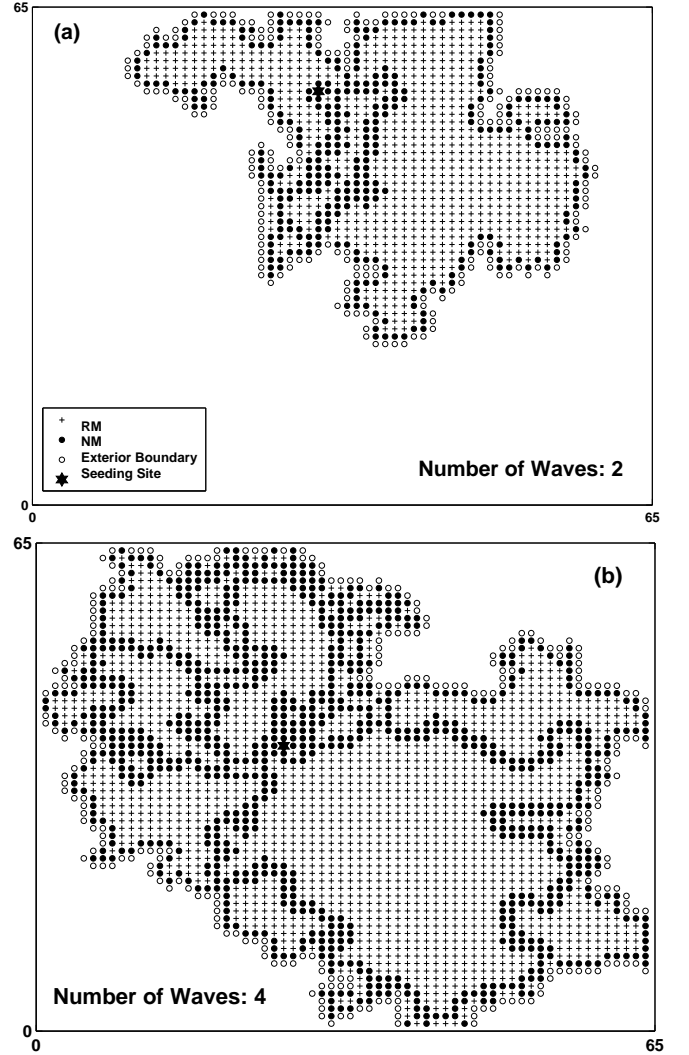


FIG. 1: (a) Fine structure and exterior boundary of a simple avalanche composed of two waves, and (b) a complex avalanche composed of 4 waves in a  $64 \times 64$  lattice. Here, the (+), (•) and (o) signs represent  $\mathcal{RM}$ ,  $\mathcal{NM}$ , and the exterior boundary of the avalanche, respectively. Note the simple pattern of the  $\mathcal{NM}$  structure in 1(a) compared to that of 1(b).

structure might be lost, and we may observe a complex, interweaved pattern of  $\mathcal{NM}$  and  $\mathcal{RM}$  structures. These two scenarios are shown in Figs. 1(a) and 1(b), respectively.

### IV. COMPLEXITY IN THE FINE STRUCTURE OF AVALANCHES

In Ref. [29] it has been shown that the boundaries of consecutive waves are not simply related to each other. In fact, the complexity present in the fine structure of an avalanche is a direct consequence of the “complex superposition” of the boundaries of successive waves making

up an avalanche. By “complex superposition” we mean that in general in an avalanche, excluding the exterior boundary,

$$n_{\mathcal{NM}} \neq \sum_k (n_{\mathcal{NM}})_{\mathcal{W}_k}. \quad (7)$$

Here and in the following we use symbols with subscript ( $\mathcal{W}$ ) to denote those quantities pertaining to a wave.  $k$  is the index of the waves making up a given avalanche. Equation (7) simply states that the non-recurrent macrostate ( $\mathcal{NM}$ ) of an avalanche is not the simple sum of the wave boundaries which form the avalanche. So, the boundaries of successive waves can interact and interpenetrate each other. Due to this interaction, the fine structure of an avalanche may show a complex pattern.

A quantity that contains valuable information about the dynamical processes underlying the formation of the fine structure of a relaxation process is the ratio of the size of  $\mathcal{NM}$  to  $\mathcal{RM}$ , i.e.  $n_{\mathcal{NM}}/n_{\mathcal{RM}}$ . In Fig. 2, we have compared the conditional expectation value of this quantity for avalanches and waves of a given area,  $E(n_{\mathcal{NM}}/n_{\mathcal{RM}}|a)$ . While for avalanches of intermediate size the ratio of  $n_{\mathcal{NM}}/n_{\mathcal{RM}}$  tends to decrease, for larger ones we observe a gradual increase in this quantity, indicating a crossover in the avalanches’ behavior, a point which we will return to later in this article. As can be seen from the figure, the conditional expectation value for the waves simply decreases with increasing area. This simply shows that the boundary to bulk ratio for waves decreases with increasing of the wave’s size, as should be expected. However, in order to understand the nature of the observed crossover phenomenon and the differences between the behavior of waves of toppling and avalanches, we must study more fundamental quantities describing dynamical properties of the critical state. From Eq. (6) we can write the probability of having an avalanche of area  $a$  as,

$$P(a) = \sum'_{n_{\mathcal{NM}}, n_{\mathcal{RM}}} P(n_{\mathcal{RM}})P(n_{\mathcal{NM}}|n_{\mathcal{RM}}), \quad (8)$$

where the summation goes over those values of  $n_{\mathcal{NM}}(\mathcal{RM})$  that fulfill Eq. (6). Here,  $P(n_{\mathcal{RM}})$  is the probability of having an avalanche with a  $\mathcal{RM}$  structure of size  $n_{\mathcal{RM}}$ .  $P(n_{\mathcal{NM}}|n_{\mathcal{RM}})$  is the conditional probability distribution function (CPDF) of having a  $\mathcal{NM}$  structure of size  $n_{\mathcal{NM}}$  for a particular value of  $n_{\mathcal{RM}}$ . This equation, through the quantity  $P(n_{\mathcal{NM}}|n_{\mathcal{RM}})$ , establishes a direct connection between properties of the critical state and dynamical aspects of the relaxation processes. In Fig. 3, we have plotted the CPDF,  $P(n_{\mathcal{NM}}|n_{\mathcal{RM}})$  for several values of  $n_{\mathcal{RM}}$ . We note that this function is asymmetric about its maximum value. However, more importantly, we observe the emergence of another local maximum with increasing  $n_{\mathcal{RM}}$  (point (b)). This suggests the emergence of a different type of behavior as  $n_{\mathcal{RM}}$  (or avalanche size) increases.

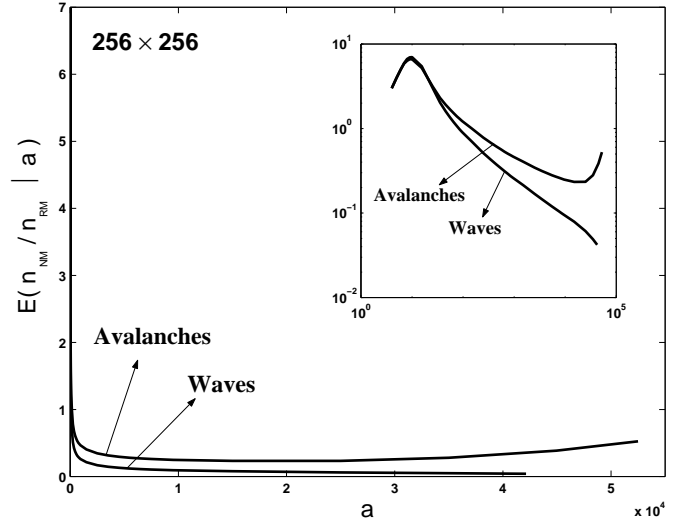


FIG. 2: Comparison between the conditional expectation value of  $n_{\mathcal{NM}}/n_{\mathcal{RM}}$  in a system of linear size  $L=256$  for avalanches and waves of a given area. To reduce the statistical fluctuation at large areas we have binned the data logarithmically. This will result in losing a fraction of large avalanches with  $E(n_{\mathcal{NM}}/n_{\mathcal{RM}}|a) \geq 1$ , which manifest themselves when one considers the unbinned data. Inset shows the same quantity on a double logarithmic graph. The bumps in the left hand side of the inset are due to the grid properties of the lattice.

What possible classification of avalanches can distinguish between the two peaks in Fig. 3, and to what extent is it related to the complexity of the fine structures of relaxation processes? If we consider a similar quantity in the simple  $\mathcal{NM}$ , like the  $\mathcal{NM}$  of waves, our simulation shows that although the asymmetric form of the CPDF persists, there is no significant change in the general form of this function, i.e. no local maximum emerges as  $n_{\mathcal{RM}}$  is increased, see Fig. 4. Therefore, in analogy with simple ( $\mathcal{NM}$ ) structure of waves, we define an avalanche to be simple, i.e. of simple fine structure, if successive waves making up that avalanche are spatially contained in each other, i.e. their  $\mathcal{NM}$  structures (boundaries) are not interpenetrating, see Fig. 1(a). In this case we have  $n_{\mathcal{NM}} \approx \sum_k (n_{\mathcal{NM}})_{\mathcal{W}_k}$ . Restrictly speaking, this definition does not exclude the possibility of a weak interaction (mixing) between the  $\mathcal{NM}$  structures of a wave and that of its predecessor, so that in general Eq. (7) holds. However, it is the extent of the violation of the equality which categorizes avalanches into simple or complex. In mathematical notation, for a sequence of waves,  $\mathcal{W}_k$ ,  $k = 1, \dots, n$ , making up an avalanche we have,

$$\text{Simple Fine Structure} \Leftrightarrow \forall i \in \mathcal{W}_k \Rightarrow i \in \mathcal{W}_{k-1}.$$

On the other hand, we consider a fine structure as complex, if the boundaries of successive waves exceed the confines of their predecessors, see Fig. 1(b).

$$\text{Complex Fine Structure} \Leftrightarrow \exists i \in \mathcal{W}_k \ni i \notin \mathcal{W}_{k-1}.$$

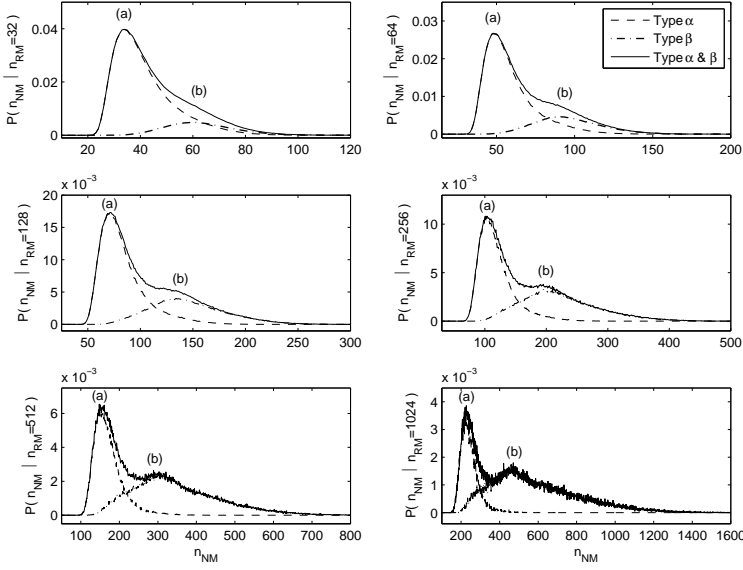


FIG. 3: Normalized CPDF  $P(n_{\mathcal{NM}}|n_{\mathcal{RM}})$  of avalanches as a function of  $n_{\mathcal{NM}}$  for different values of  $n_{\mathcal{RM}}$  on a  $512 \times 512$  lattice. Note the emergence of a local maximum with increasing  $n_{\mathcal{RM}}$ . This indicates the emergence of a different sort of behavior with increasing avalanche size. Here the dashed and dash-dotted lines show the decomposition of the CPDF to that of type  $\alpha$  and type  $\beta$  avalanches.

At this point, we need to make a connection between the properties of the critical state and those of the type  $\alpha$  and type  $\beta$  avalanches. Let  $N$  be the total number of avalanches. Then,  $N = N_\alpha + N_\beta$  where  $N_\alpha$  and  $N_\beta$  are the number of type  $\alpha$  and type  $\beta$  avalanches, respectively. Using the abbreviated notations  $n_{\mathcal{NM}} \mapsto y$  and  $n_{\mathcal{RM}} \mapsto x$ , we have the following definitions:

- $N^{\alpha(\beta)}$ : Total number of type  $\alpha$  (or  $\beta$ ) avalanches.
- $N_{yx}^{\alpha(\beta)}$ : Total number of type  $\alpha$  (or  $\beta$ ) avalanches with  $\mathcal{NM}$  of size  $y$  and  $\mathcal{RM}$  of size  $x$ .
- $N_y^{\alpha(\beta)}$ : Total number of type  $\alpha$  (or  $\beta$ ) avalanches with  $\mathcal{NM}$  of size  $y$ .
- $N_x^{\alpha(\beta)}$ : Total number of type  $\alpha$  (or  $\beta$ ) avalanches with  $\mathcal{RM}$  of size  $x$ .
- $N_x$ : Total number of avalanches with  $\mathcal{RM}$  of size  $x$ .
- $N_{yx}$ : Total number of avalanches with  $\mathcal{NM}$  of size  $y$  and  $\mathcal{RM}$  of size  $x$ .

Using these definitions, we can rewrite Eq. (8) in terms of the properties of type  $\alpha$  and type  $\beta$  avalanches. We start with:

$$\begin{aligned} P(y|x) &= \frac{N_{yx}}{N_x} = \frac{N_{yx}^\alpha + N_{yx}^\beta}{N_x} = \frac{N_{yx}^\alpha}{N_x} \frac{N_x^\alpha}{N_x} + \frac{N_{yx}^\beta}{N_x} \frac{N_x^\beta}{N_x} \\ &= P_\alpha(y|x) \frac{N_x^\alpha}{N_x} + P_\beta(y|x) \frac{N_x^\beta}{N_x}. \end{aligned} \quad (9)$$

Substituting Eq. (9) in (8), and using the fact that  $P(x) = N_x/N$  we have,

$$P(a) = \sum_{x, y} P(x) P(y|x) = \sum_{x, y} \left[ P_\alpha(y|x) \frac{N_x^\alpha}{N} + P_\beta(y|x) \frac{N_x^\beta}{N} \right], \quad (10)$$

where the summation goes over those values of  $x$  and  $y$  that fulfill Eq. (6). Finally, using the relation,

$$N_x^{\alpha(\beta)}/N = \frac{N_x^{\alpha(\beta)}}{N^{\alpha(\beta)}} \frac{N^{\alpha(\beta)}}{N} = P_{\alpha(\beta)} P_{\alpha(\beta)}(x),$$

we have,

$$P(a) = \sum_{x, y} [P_\alpha P_\alpha(x) P_\alpha(y|x) + P_\beta P_\beta(x) P_\beta(y|x)]. \quad (11)$$

According to Eq. (11), an avalanche size distribution function can be written as a separate combination of type- $\alpha$  and type- $\beta$  distribution functions. Now, if type- $\alpha$  and type- $\beta$  avalanches have similar scaling properties, one can expect the avalanche-size probability distribution functions to scale accordingly. However, if these two types of avalanches have different and distinct scaling properties, one cannot find a consistent scaling behavior

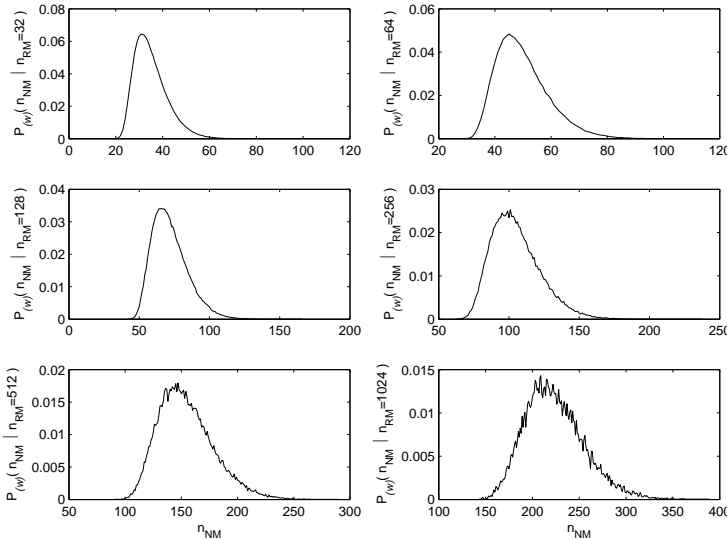


FIG. 4: Normalized CPDF  $P_W(n_{\mathcal{NM}}|n_{\mathcal{RM}})$  of waves as a function of  $n_{\mathcal{NM}}$  for different values of  $n_{\mathcal{RM}}$ . No local maximum emerges as  $n_{\mathcal{RM}}$  is increased.

We call these two classes of avalanches type  $\alpha$  and type  $\beta$ . As shown in Fig. 3, our numerical simulation indicates that the CPDF  $P(n_{\mathcal{NM}}|n_{\mathcal{RM}})$  is a superposition of CPDF's of type- $\alpha$  and type- $\beta$  avalanches, in which the first and second maxima correspond to type  $\alpha$  (Fig. 3, point (a)), and type- $\beta$  avalanches (Fig. 3, point (b)), respectively.

for the overall probability distribution function. This is in fact the key message of the present Article. In the next section we will give analytical as well as numerical evidence on how the scaling properties of these two types of avalanches differ from each other. The key difference, as we will see, is the size-dependence of the scaling behavior in type- $\beta$  avalanches.

## V. SCALING OF CPDF FOR DIFFERENT TYPES OF AVALANCHES

In order to investigate the properties of  $P(a)$  it is important to study the properties of  $P_\alpha(y|x)$  and  $P_\beta(y|x)$ . We have carried out an extensive study of such CPDF's. We find that the scaling behavior of these distributions are distinctly different. Accordingly, we propose the following scaling ansatz for the above distribution functions:

- Type  $\alpha$  avalanches:

$$P_\alpha(y|x) = x^{-\gamma_\alpha} U_\alpha\left(\frac{y - E_\alpha(y|x)}{x^{\gamma_\alpha}}\right). \quad (12)$$

- Type  $\beta$  avalanches:

$$P_\beta(y|x) = x^{-\gamma_\beta} U_\beta\left(\frac{y + x + E_\beta(y|x)}{x^{\gamma_\beta}}\right), \quad (13)$$

where  $\gamma_\beta$  is a size-dependent exponent, i.e.  $\gamma_\beta = \gamma_\beta(x)$ . Here  $U_\alpha$  and  $U_\beta$  are universal functions, and  $E_{\alpha(\beta)}(y|x)$  is the mean value of the given CPDF, defined through the relation  $E_{\alpha(\beta)}(y|x) = \int y P_{\alpha(\beta)}(y|x) dy$ .

From Eqs. (12) and (13), we can readily calculate the scaling behavior of the first and second moments of  $y$ . This provides a suitable way via which one can confirm the proposed scaling ansatz for the corresponding CPDF's.

Let us first consider the case of type  $\beta$  avalanches. Using the suggested form in Eq. (13), we have

$$\begin{aligned} E_\beta(y|x) &= \int_0^\infty y P_\beta(y|x) dy \\ &= x^{-\gamma_\beta} \int_0^\infty y U_\beta((y + x + E_\beta(y|x))/x^{\gamma_\beta}) dy. \end{aligned} \quad (14)$$

Performing the change of variable  $z = y + x + E_\beta(y|x)$  in Eq. (14) and integrating we obtain,

$$E_\beta(y|x) = \frac{1}{2} C x^{\gamma_\beta} - \frac{x}{2}, \quad (15)$$

where  $C = \int \xi U_\beta(\xi) d\xi$ ;  $\xi = z x^{-\gamma_\beta}$ . So, after the addition of the linear term  $x/2$  to  $E_\beta(y|x)$ , it must scale as  $x^{\gamma_\beta(x)}$  for different scaling regions defined by the area or the size of the  $\mathcal{RM}$  structure of an avalanche. In the case of type  $\alpha$  avalanches, we cannot obtain the scaling behavior of the corresponding conditional expectation value,  $E_\alpha(y|x)$  from  $P_\alpha(y|x)$ , as we did in the case of type  $\beta$

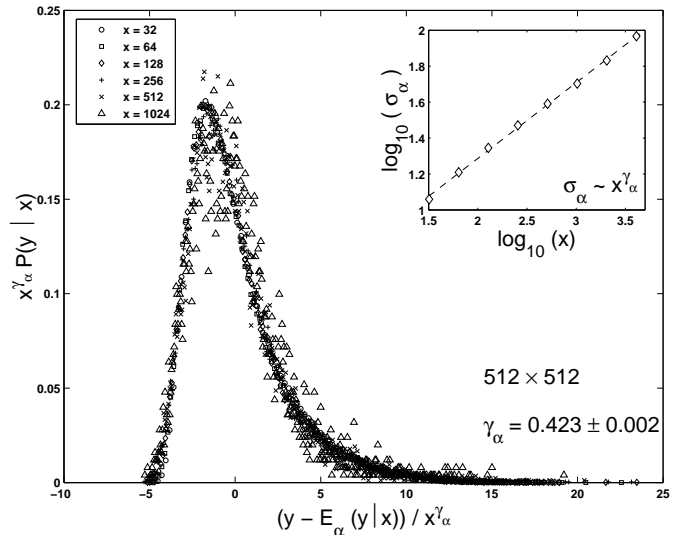


FIG. 5: CPDF data collapse using the suggested form, Eq. (12), for type- $\alpha$  avalanches. Here, we obtain a good collapse for different values of  $n_{\mathcal{RM}}$  with  $\gamma_\alpha = 0.423 \pm 0.002$ . Inset shows the scaling of  $\sigma_\alpha$  with  $x$  for different values of  $x$  ranging from  $x = 32$  to  $x = 4096$ . The slope of the dashed line is equal to  $\gamma_\alpha = 0.423$

avalanches. So, to obtain any further information, we must look at higher moments of  $y$ , e.g.  $E_\alpha(y^2|x)$ .

$$\begin{aligned} E_\alpha(y^2|x) &= \int_0^\infty y^2 P_\alpha(y|x) dy \\ &= x^{-\gamma_\alpha} \int_0^\infty y^2 U_\alpha((y - E_\alpha(y|x))/x^{\gamma_\alpha}) dy. \end{aligned} \quad (16)$$

Performing the change of variable  $z = y - E_\alpha(y|x)$ , we can rewrite Eq. (16) as,

$$E_\alpha(y^2|x) = C x^{2\gamma_\alpha} + [E_\alpha(y|x)]^2, \quad (17)$$

where  $C = \int \xi^2 U_\alpha(\xi) d\xi$ ;  $\xi = z x^{-\gamma_\alpha}$ . Therefore,

$$\sigma_\alpha = [E_\alpha(y^2|x) - (E_\alpha(y|x))^2]^{1/2} \sim x^{\gamma_\alpha}, \quad (18)$$

where  $\sigma_\alpha$  is the standard deviation of the given distribution. In obtaining the last relation we have used the fact that  $\int \xi U_\alpha(\xi) d\xi = 0$ . So, in the case of type  $\alpha$  avalanches  $\sigma_\alpha$  scales as  $x^{\gamma_\alpha}$ .

To verify the proposed scaling forms and also to extract the scaling exponents  $\gamma_\alpha$  and  $\gamma_\beta$ , we implement the method of data collapse in line with Ref. [25]. In Fig. 5, we have applied this technique to type  $\alpha$  avalanches for different values of  $x (= n_{\mathcal{RM}})$ . As can be seen, we obtain a reasonable collapse with the scaling exponent  $\gamma_\alpha = 0.423$ . The inset shows the scaling of standard deviation,  $\sigma_\alpha$ , with  $x$  for the corresponding CPDF's. There the dashed line shows  $x^{\gamma_\alpha}$ . This verifies in a straightforward manner, the validity of Eq. (18), thus lending support to our scaling ansatz, Eq.(12). By performing a similar analysis on an ensemble of waves, we could verify that the CPDF  $P_{\mathcal{W}}(y|x)$  of waves of toppling also possesses a similar scaling to type  $\alpha$  avalanches, Eq. (12),

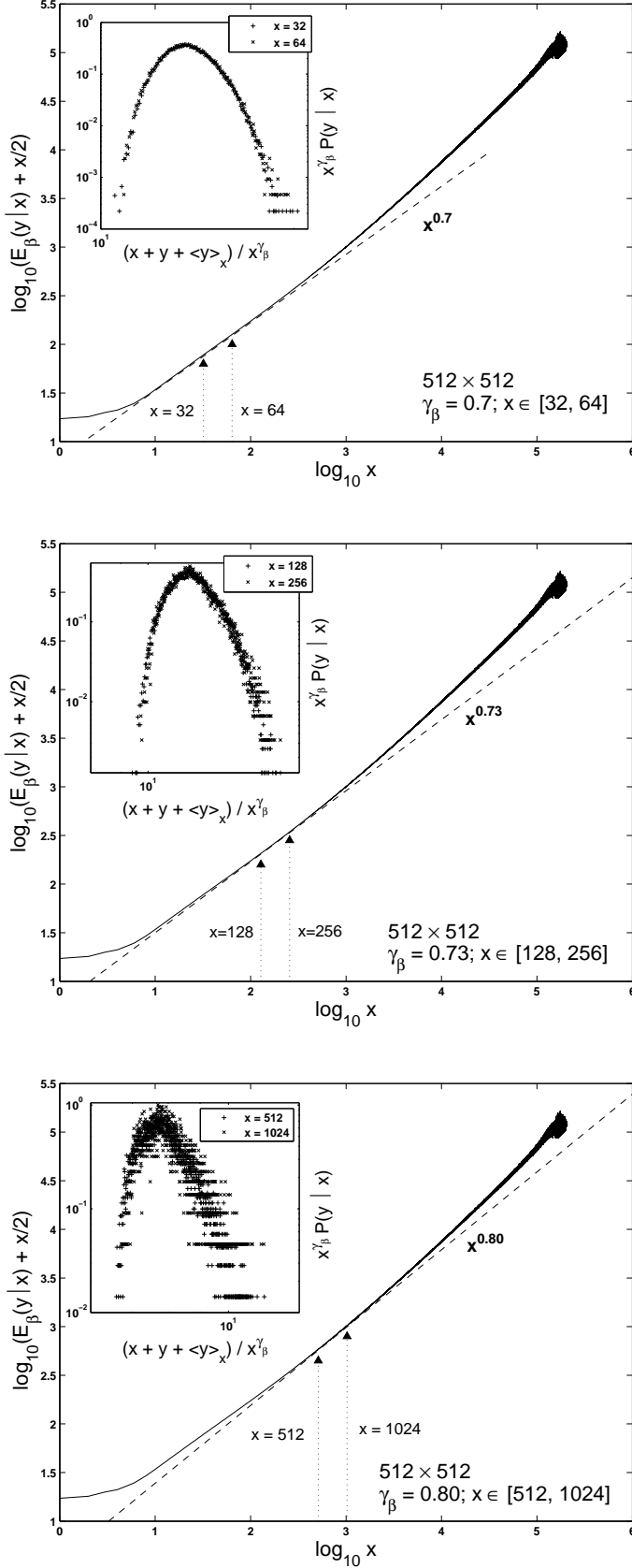


FIG. 6: Scaling behavior of the conditional expectation value,  $E_\beta(y|x)$ , after the addition of the linear term  $x/2$  in a system of linear size  $L = 512$  for (a) small avalanches ( $32 \leq x \leq 64$ ), (b) intermediate avalanches ( $128 \leq x \leq 256$ ), (c) and large avalanches ( $512 \leq x \leq 1024$ ). Insets show the collapse of CPDF,  $P_\beta(y|x)$ , using the suggested form, Eq. (13), for different scaling regions, which determines the value of the scaling exponent  $\gamma_\beta(x)$ . Note how  $\gamma_\beta$  increases with avalanche size.

with the scaling exponent  $\gamma_{(\mathcal{W})} = 0.47$ . This indicates an interesting similarity between the scaling behavior of these two types of relaxation events. However, the obvious difference in the values of the scaling exponents,  $\gamma_\alpha$  and  $\gamma_{(\mathcal{W})}$ , reveals the essential differences in the nature of these two relaxation processes. In other words, one cannot consider type- $\alpha$  avalanches as a simple sum of waves making up that avalanche.

Now, the situation is not as simple for type- $\beta$  avalanches. We observe that we cannot find a unique exponent  $\gamma_\beta$ , which collapses our data for all values of  $x (= n_{\mathcal{RM}})$ . Instead, we find that  $E_\beta(y|x) + x/2$  does not have a unique slope (on a logarithmic plot) and shows an obvious curvature as can be seen in Fig. 6. Here instead, we find different scaling exponents for different scaling regions. We divide our scaling region into small, intermediate, and large avalanches and perform our collapse (the insets in Fig. 6) with different exponent for each of these regions. This is shown in different parts of Fig. 6. The important and fundamental difference here in the case of  $\beta$ -avalanches is that  $\gamma_\beta$  is size-dependent and increases with avalanche's size. However, this increase cannot be unbound and  $\gamma_\beta$  should eventually saturate. Our numerical results show that this exponents saturates at  $\gamma_\beta = 1$ , indicating a range of  $0.7 \leq \gamma_\beta \leq 1.0$  for this exponent, see Fig. 7. Since we do not expect that the boundary of a relaxation process grows larger than its bulk, the value  $\gamma_\beta(x \gg 1) = 1$  is in fact the physical upper limit for this exponent. Another way to see that  $\gamma_\beta$  saturates at  $\gamma_\beta = 1.0$  is to plot  $E_\beta(\frac{y}{x}|x) + 1/2$ , which according to our scaling ansatz should scale as  $x^{\gamma_\beta(x)-1}$ . This is shown as an inset in Fig. 7. One can see this saturation as the eventual  $x$ -independence of the plot for large avalanches. We therefore conclude that  $\gamma_\beta$ , unlike  $\gamma_\alpha$ , does not have a fixed value and in fact varies between 0.7 and 1.0 with increasing avalanche size. Such form of avalanche-size dependent exponent is a result of the complexity inherent in the dynamical behavior of type- $\beta$  avalanches.

It is now clear why simple FSS fails in the 2-D BTW ASM. Avalanches can be categorized in two classes each of which has distinctly different scaling properties. The combination of these two cannot exhibit a consistent scaling behavior. Moreover, the culprit is identified as type- $\beta$  avalanches for which the scaling exponent depends on the avalanche size. The coexistence of two different types of avalanches in the critical state, with distinctly different scaling behavior, and the failure of FSS picture, has been already observed and analytically proved in 1-D ASM [32]. However, and to the best of our knowledge, this is for the first time that this phenomenon is reported and numerically verified in the 2-D BTW ASM. Note that this classification is irrespective of the boundary and system size effects and is inherent in the dynamical properties of an avalanche.

Here, one might wonder if our distinction of type- $\alpha$  and  $-\beta$  avalanches is merely an analysis of avalanches made up of a simple wave versus a collection of waves. In order to address this issue we have performed the same

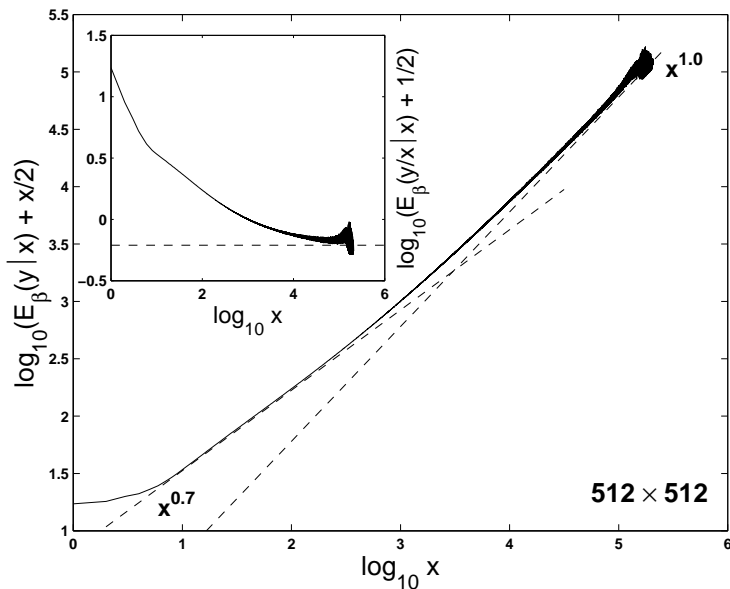


FIG. 7: Conditional expectation value,  $E_\beta(y|x)$ , after the addition of the linear term  $x/2$  for a system of linear size  $L = 512$ . Note the curvature in the graph for different values of  $x$ . Here the dashed lines have slopes 0.7 and 1.0 respectively. Inset shows the scaling behavior of  $E_\beta(\frac{y}{x}|x) + 1/2$  for different values of  $x$ . The gradual increase of  $\gamma_\beta(x)$  to the value  $\gamma_\beta(x) = 1$  is clear. The horizontal dashed line is as a reference for sight.

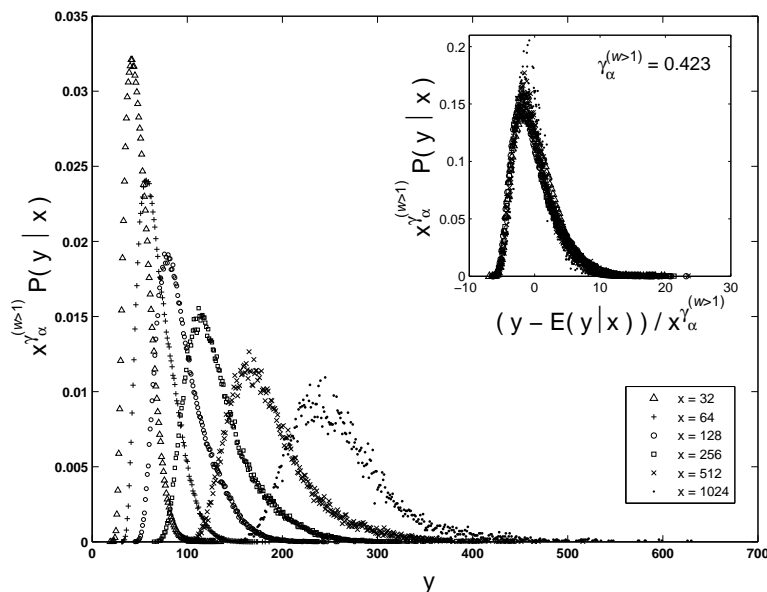


FIG. 8: CPDF of type  $\alpha$  avalanches with more than one wave for different values of  $n_{\mathcal{RM}}$  in a  $512 \times 512$  lattice. Inset shows the collapse of CPDF of these avalanches using the suggested form of the CPDF of type  $\alpha$  avalanches. Here again one obtains a reasonable collapse with  $\gamma_\alpha^{(w>1)} = 0.423$ .

analysis on the properties of type- $\alpha$  avalanches made up of more than one wave. Our numerical analysis shows that, despite the existence of multiple toppling events (waves) in this class of avalanches, there is no dependency in their scaling behavior on the  $\mathcal{NM}$  structure (or area) of the avalanches. In fact, as it is shown in Fig. 8, this type of avalanches collapse with the same exponent as that of type- $\alpha$  avalanches shown in Fig. 5. We now can assert that what distinguishes  $\alpha$ - and  $\beta$ -avalanches is the interaction (mixing) of wave boundaries and not the mere existence of multiple waves. In other words, simple waves make up avalanches of type- $\alpha$  and complex (mixing) waves constitute complex or  $\beta$ -avalanches.

Finally, of particular importance is the relation between the behavior of the scaling exponents  $\gamma_{\alpha(\beta)}$  and the deviation from simple power law behavior in the quantities such as  $P(s)$ , where  $s$  is the size (or total number of topplings) in an avalanche, in the 2-D BTW model. In order to gain a better understanding of this general behavior, we have also looked at some other SOC models in this regard. Our preliminary results in the case of Manna model, which obeys FSS, show that the scaling exponent  $\gamma$  maintains a constant value close to 0.5 [31, 33, 34]. Such observation, as well as the similarities between the behavior of waves and type- $\alpha$  avalanches, makes us believe that the constancy of the  $\gamma$  exponent indicates that the scaling behavior of the corresponding relaxation process follows a simple power law behavior. In the case of size dependency of the exponent  $\gamma$ , it is the rate of change of  $\gamma$  which determines whether the power law behavior emerges or not. For example, in the particular case of the type- $\beta$  avalanches in the present model, as  $\gamma_\beta$  approaches unity for large avalanches, the rate of change of  $\gamma_\beta$  becomes very small, as can be seen from Figs. 6 and 7. Therefore, we expect that the power law behavior is recovered for large type- $\beta$  avalanches. While this reasoning holds true for quantities like  $P(s)$  in which multiple toppling events play an important role, one should take much more care in interpreting the behavior of other quantities such as  $P(a)$ . In fact, due to the large fluctuations in the size of a  $\mathcal{NM}$  structure of a  $\beta$ -avalanche (see Fig. 3), the delicate size dependency of the exponent  $\gamma_\beta$  is effectively blurred by performing the summation in Eq. (11). In such situation, the effect of averaging, as well as ignoring the fundamental differences between the two types of avalanches, will result in a seemingly power law behavior in the case of  $P(a)$  [23]. Indeed, by gathering separate statistics for  $\alpha$ - and  $\beta$ -avalanches, one can clearly observe how the scaling behavior of quantities such as  $P_\beta(a)$  and  $P_\beta(s)$  deviates from a simple power law for small and intermediate avalanche sizes. However, the power law behavior is recovered for large  $\beta$ -avalanches [31].

## VI. CONCLUSION

To summarize, in this Article, we have shown that avalanches in the ASM can have complex fine structures



as a result of interaction of wave boundaries within a given avalanche. We used these interactions to define simple (type- $\alpha$ ) and complex (type- $\beta$ ) avalanches. We have studied the scaling behavior of these two avalanche types in detail and have highlighted their differences. We have shown how one can view the general dynamical scaling properties of this model in terms of scaling properties of the combined type- $\alpha$  and type- $\beta$  avalanches. We have proposed scaling ansatz for these two types of avalanches and have verified them numerically, thus showing how these two types of avalanches have distinctly different scaling behavior. In particular, while type- $\alpha$  avalanches are characterized by a constant-value scaling exponent, type- $\beta$  avalanches are characterized by an avalanche-size dependent exponent. We believe, this distinction between type- $\alpha$  and type- $\beta$  avalanches underlies the failure of consistent (finite-size) scaling in this model. Moreover, we argued how the size dependency of the scaling exponent in  $\beta$ -avalanches leads to the deviation from power law behavior in important quantities describing the dynamical behavior of the avalanches, such as  $P(s)$ . In addition, our results indicate that due to the coexistence of two distinctly different relaxation events in the critical

state of the BTW model, one must separate these two events in gathering any reliable statistics from the system. We hope that our analysis is helpful in answering some of the long standing problems on the behavior of the prototype model of SOC in two dimensions. Moreover, we note that our classification of avalanches opens up many questions as well. For example, how are the dynamical behavior of these avalanches different from each other? In particular, what are the essential characteristics of wave boundary interactions? Can such classifications be useful in other models of SOC? We plan to address some of these issues in a forthcoming publication [31].

## VII. ACKNOWLEDGEMENT

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- [1] P. Bak, C. Tang, and K. Wiesenfeld, Phys. Rev. Lett. **59**, 381 (1987).
  - [2] For a review of the subject and recent advances see: P. Bak, *How Nature Works: The Science of Self-Organized Criticality* (Copernicus, New York, 1996); H. J. Jensen, *Self-Organized Criticality: Emergent Complex Behavior in Physical and Biological Systems* (Cambridge University Press, 1998); K. Christensen and N. R. Moloney, *Complexity and Criticality* (Imperial College Press, Advanced Physics Texts, Vol. 1, 2005); M. Alava, *Self-Organized Criticality as a Phase Transition*, arXiv: cond-mat/0307688.
  - [3] P. Bak, C. Tang, and K. Wiesenfeld, Phys. Rev. A **59**, 364 (1988);
  - [4] V. Frette, K. Christensen, A. Malthes-Srenssen, J. Feder, T. Jssang, and P. Meakin, Nature **49**, 379 (1996).
  - [5] S. Field, J. Witt, and F. Nori, Phys. Rev. Lett. **74**, 1206 (1995).
  - [6] D. Raup, Science, **231**, 1528 (1986); T. Krink, R. Thomsen, *Self-organized criticality and mass extinction in evolutionary algorithms.*, Proc. IEEE Int. Conf. on Evolutionary Computation, 1155-1161 (2001).
  - [7] D. Hughes, M. Paczuski, R. O. Dendy, P. Helander, and K. G. McClements, Phys. Rev. Lett. **90**, 131101 (2003).
  - [8] K. Christensen, *Self-Organized Criticality in Models of Sandpiles, Earthquakes, and Flashing Fireflies*, Ph.D. Thesis, Department of Physics, University of Århus, Denmark (1992).
  - [9] L. P. Kadanoff, S. R. Nagel, L. Wu, S. Zhou, Phys. Rev. A **39**, 6524 (1989); Z. Olami, H. J. S. Feder, and K. Christensen, Phys. Rev. Lett. **68**, 1244 (1992); K. Christensen, A. Corral, V. Frette, J. Feder, and T. Joessang, Phys. Rev. Lett. **77**, 107 (1996); S. S. Manna, J. Phys. A **24**, L363 (1991).
  - [10] D. Dhar, Phys. Rev. Lett. **64**, 1613 (1990).
  - [11] J. M. Carlson, E. R. Grannan, C. Singh, G. H. Swindle, Phys. Rev. E **48**, 688 (1993).
  - [12] A. Montakhab, J. M. Carlson, Phys. Rev. E **58**, 5608 (1998).
  - [13] L. Pietronero, A. Vespignani, and S. Zapperi, Phys. Rev. Lett. **72**, 1690 (1994).
  - [14] V. B. Priezzhev, D. V. Ktitarov, and E. V. Ivashkevich, Phys. Rev. Lett. **76**, 2093 (1996).
  - [15] E. V. Ivashkevich, Phys. Rev. Lett. **76**, 3368 (1996).
  - [16] S. N. Majumdar and D. Dhar, J. Phys. A **24**, L357 (1991).
  - [17] V. B. Priezzhev, J. Stat. Phys. **74**, 955 (1994).
  - [18] E. V. Ivashkevich, J. Phys. A **27**, 3643 (1994).
  - [19] D. Dhar, Physica A, **369**, 27 (2006).
  - [20] C. Tang and P. Bak, Phys. Rev. Lett. **60**, 2347 (1988).
  - [21] N. Goldenfeld, *Lectures on Phase Transitions and the Renormalization Group*, (Addison-Wesley, 1992), Frontiers in Physics, Vol. 85.
  - [22] M. De Menech, A. L. Stella, and C. Tebaldi, Phys. Rev. E **58**, 2677 (1998).
  - [23] C. Tebaldi, M. De Menech, and A. L. Stella, Phys. Rev. Lett. **83**, 3952 (1999).
  - [24] S. Lübeck and K. D. Usadel, Phys. Rev. E **55**, 4095 (1997).
  - [25] A. Chessa, H. E. Stanley, A. Vespignani, and S. Zapperi, Phys. Rev. E **59**, R12 (1999).
  - [26] B. Drossel, Phys. Rev. E **61**, R2168 (2000).
  - [27] D. V. Ktitarov, S. Lübeck, P. Grassberger, and V. B. Priezzhev, Phys. Rev. E **61**, 81 (2000).
  - [28] E. V. Ivashkevich, D. V. Ktitarov, and V. B. Priezzhev, Physica A **209**, 347 (1994).
  - [29] M. Paczuski and S. Boettcher, Phys. Rev. E **56**, R3745 (1997).

- [30] M. De Menech, and A. L. Stella, Phys. Rev. E, **62**, R4528 (2000).
- [31] A. Abdolvand and A. Montakhab, (in preparation).
- [32] A. A. Ali and D. Dhar, Phys. Rev. E **52**, 4804 (1995).
- [33] S. S. Manna, J. Phys. A **24**, L363 (1991).
- [34] S. Lübeck, Phys. Rev. E **61**, 204 (2000).